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ON GENERAL PURPOSE PROGRAMS FOR FINITE ELEMENT ANALYSIS, WITH SPECIAL REFERENCE TO GEOMETRIC AND MATERIAL NONLINEARITIES

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by

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1. Structural mechanics

Abstract

A summary of the state-of-the-art in nonlinear finite element analysis is made by describing a nonlinear theory and presenting some case studies. The formulation is applicable to problems of large displacement and small strains.

The paper then focuses on the general purpose program. The concept and development of a general purpose program is described. A discussion is then made of the different sizes of problems which can be solved by such a program. These sizes are dependent on the available computer core. The conclusion is made that the general purpose program is a powerful means of implementing finite element analysis over a wide spectrum of problems in structural mechanics.

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Introduction

In recent years we have seen an increasing use of the finite element method for both research and development. This paper briefly summarizes the method and traces the reasons for its widespread use.

The finite element method is dependent on the combination of two basic ideas. The first is the recognition that problems in continuum mechanics may be solved by complete satisfaction of only one of the two requirements of equilibrium and compatibility if the other condition is also satisfied in an integral sense. This approximate solution of the remaining condition in the integral sense is brought about by the use of the principle of virtual work and theorems resulting from it. The second idea is that the function for a whole domain may be better approximated by local functions assumed within subdomains which also maintain continuity of the functions across the subdomains. The undetermined parameters for the assumed local subdomain functions are then related to physical quantities of displacement [1] or force [2] degrees of freedom at points or nodes on the boundaries of the subdomain. This then allows the definition of equations which define either stiffness or flexibility matrices for the subdomain (or discrete element). The combination of the relaxation of the requirement of either equilibrium or compatibility and the localized functions whose unknowns are represented by quantities at nodes results in a considerable easing of the problems of geometry and boundary conditions. Different subdomains may be modeled with different functions and these may be used simultaneously for an analysis.

The finite element theory is usually cast in matrix theory since this allows the large background of matrix theory to be exploited. Its development occurred at the same time as the order of magnitude increase in computer speeds and core size. This happy confluence of all the factors discussed above has

given rise to the widespread development and use of the finite element method. Initially, its implementation took place in the form of specialized programs written for specific purposes. Then as the method developed it became obvious that a more general approach could be adopted in which the common tasks for every finite element could be programmed once and for all. This has resulted in the development of the general purpose finite element programs. These efforts have the same overall strategy as the SPADE projects adopted for partial differential equations. At present, the continued development of the general purpose programs appears to be the best means of implementing finite element theory. Yet notwithstanding this, little has appeared in the literature which specifically concerns itself with the features and underlying philosophy of the general purpose program. It is the purpose of the current paper to discuss the development of a general purpose program.

Review of Literature

The present paper will trace developments from the original paper by Turner et al. [1], and attention will be confined solely to the direct stiffness method of finite element analysis.

Initially, work was concerned with developing elements [3-8] with compatible displacements at the boundaries. This phase can now be said to be complete, and elements exist to cover any two- or three-dimensional solids (including shell structures). We may classify elements by the type of displacement modes assumed and with this classification three types of elements can be recognized. In its two-dimensional form these three may be referred to as the triangular, the orthogonal and the piecewise patching type of element.

In the triangular type of element, the displacement modes are assumed to take the form of complete polynomials [1-4]. In the orthogonal type of element,

the displacement modes are assumed to take the form of either Lagrange Polynomials [3,5] or Hermitian Polynomials [6]. The Lagrange Polynomials [5] are also used to extend the element formulation to quadrilaterals and curve shaped elements by isoparametric techniques. In the patching type of element [7,8], usually used for shells and referred to as the DeVeubeke element, the displacement modes are assumed to be made up of a compatible patching of complete polynomials. Three-dimensional equivalents also exist for the first two types of elements.

At the same time work [9,10] was reported which established a framework by which the finite element method could be related to the methods used in continuum mechanics. The finite element method is now recognized as a special case of the Rayleigh-Ritz method where generalized modes are assumed over subdomains where the generalized modes give rise to inter-element compatibility of displacements.

With the establishment of the method, attention was turned to extensions for nonlinear analysis. In the area of material nonlinearity, two methods were developed for elastic-plastic analysis. The method of initial strains is based on the idea of modifying the equations of equilibrium so that the elastic equations can be used throughout on the left-hand side of the equations. Modifications are introduced on the right-hand side of the equation to compensate for the fact that the plastic strains do not cause any change in the stresses. On the other hand, the tangent modulus method is based on the linearity of the incremental laws of plasticity and approaches the problem in a piecewise linear fashion. The load is applied in increments, and at each increment a new set of coefficients is obtained for the equilibrium equations. The matrix equations for the finite element analysis using the method of initial strains were developed by Padlog et al. [11], Argyris et al. [12] and Jensen et al. [13]. A recent paper by Witmer [14] summarizes the latest application of the method. The equations

for the tangent modulus method were developed by Pope [15], Swedlow and Yang [16] and Marcal and King [17]. The two methods were compared by Marcal [18] and a close similarity was found between them.

Progress has also been made in the area of geometric nonlinearity. Large displacement analysis by the finite element method was first proposed by Turner et al. [19]. Initial stress stiffness matrices were developed to account for the effect of initial stress in truss and plane stress assemblies. Subsequent work on the derivation of the initial stress matrices for other elements were reported by Argyris et al. [20], Gallagher et al. [21] and Kapur and Hartz [22]. Martin [23] placed the derivation of the initial stress matrix on a firm foundation by using a potential energy formulation together with the nonlinear strain displacement relation (for Green's strain). The above papers were concerned with forming matrices which account for geometric changes in the solid during an increment of load. These matrices were then either used in a piecewise linear manner or used in an eigenvalue analysis of the Euler type. Recent papers [24,25] have drawn attention to the fact that certain important terms were neglected in finite element large displacement analysis. These neglected terms result in what was called the initial displacement matrix and is a result of the coupling between the quadratic and the linear terms in the strain displacement expressions.

Other workers solved the nonlinear equations of the finite element method directly. Bogner et al. [26] performed a direct minimization of the potential energy without explicitly forming the matrix stiffness equations. The large displacement behavior was followed into the post-buckling region. Mallett and Berke [27] applied this method to frameworks and Bogner et al. to plates and shells [28]. Oden [29] and Oden and Kubitza [30] developed nonlinear stiffness relations for the nonlinear elasticity problem. The equations were solved by a

Newton-Raphson method. This series of works is perhaps best placed in a separate category. It concerns itself with large strain, large displacement analysis. The other papers reviewed previously have all implicitly assumed a large displacement small strain theory. In addition, the constitutive equations used there are in terms of an energy potential which is appropriate for a rubber-like material. A similar formulation with appropriate assumptions of constitutive behavior gives rise to equations for large strain large displacement analysis of metal structures [31]. A recent survey by Oden [32] brings out the very general nature of the finite element formulation.

Hence, we have seen that progress has been made in both nonlinear material and geometric behavior. The two nonlinear formulations do not depend on each other so that they may be profitably combined. In the present paper we shall restrict our attention to a small strain large displacement theory appropriate to shells and other solid metal structures.

In the area of elastic analysis by the finite element method, general purpose computer programs exist which are written with a view to covering the whole area of stress analysis. These general purpose programs exploit the generality of the matrix formulation of the finite element method. The programs have a library of elements which can be used for the modeling of most structures in service. Melosh et al. [33] have summarized the more recent general purpose programs.

Technical Considerations

The theory outlined here has been developed previously in [34]. It is included here for completeness. The displacement method of finite element analysis will be used throughout. The structure to be analyzed is divided into a number of elements. The behavior of each element is lumped into a number of nodal point

displacements. Conforming displacement modes or simply conforming elements are modes which maintain displacement compatibility between adjacent elements at the element boundaries. The principle of virtual work is then used to effect the lumping of the equivalent forces. The principle of virtual work is of course applicable to large displacement as well as nonlinear material behavior.

A brief outline of the method is now given. A displacement mode is first chosen for the type of element being used,

$$u = \sum_{i=1}^{n} a_i f_i(x) = [f(x)]\{a\} \tag{1}$$

where u is the displacement at position x

x is used to represent the coordinates of the element

a_i are the generalized displacements (also written $\{a\}$)

n is the number of terms in the summation.

By substituting for x at the nodes obtain

$$\{a\} = [\alpha]\{u\} \tag{2}$$

where $\{u\}$ is the displacement at the nodal points (note that there is a distinction between the bracketed and unbracketed u).

$[\alpha]$ is the nodal point to generalized displacement transformation matrix.

By the assumption of small strains we also have

$$\Delta\{a\} = [\alpha]\Delta\{u\} \tag{3}$$

where the prefix Δ denotes an increment of the quantity immediately following it.

We now define the so-called differential operator $[B]$ which transforms an increment of generalized displacement to an increment of strain.

$$\Delta\{e\} = [B]\Delta\{a\} \tag{4}$$

The differential operator [B] is a function of position x and of current displacement u . It is defined by writing the nonlinear strain displacement equations (Green's strain) in incremental form.

The strain increment can be written as a stress increment for an elastic-plastic material in the manner of Marcal and King for solids [17] and Marcal [35] for plates and shells.

$$\Delta\{\sigma\} = [D]\Delta\{e\} \tag{5}$$

The stress increments $\Delta\{\sigma\}$ are accumulated at representative points within each element. Each representative point is given a set of reference axes which deform with the element and so take the same direction as that defined by an increment of Green's strain. Thus stresses and strain increments are automatically aligned and the nonlinear equations of equilibrium can be set up with ease.

We now use the principle of virtual work to define equivalent forces {P} at the nodes for a virtual displacement $\delta\lfloor u\rfloor$.

$$\Delta\lfloor u\rfloor\{P\} = \int_V \delta\lfloor e\rfloor\{\sigma\}dV = \int_V \delta\lfloor u\rfloor[\alpha]^T[B]^T\{\sigma\}dV \tag{6}$$

where $\lfloor\ \rfloor$ denotes a row vector, and integration is performed over the volume V .

Cancelling the non-zero virtual displacements from both sides and writing equation (6) in incremental form with the aid of equation (5), obtain

$$\Delta\{P\} = \int_V [\alpha]^T\Delta[B]^T\{\sigma\}dV + \int_V [\alpha]^T[B]^T[D][B][\alpha]dV\Delta\{u\} \tag{7}$$

The last term on the right of equation (7) can be divided into a matrix which is dependent on the current displacement and one which is not. With some rearrangement, we obtain the element stiffness matrices

$$\Delta\{P\} = \left([k^{(1)}] + [k^{(2)}] + [k^{(0)}]\right)\Delta\{u\} \tag{8}$$

where $[k^{(1)}]$ is the initial stress matrix and is obtained from the first term on the right of equation (7).

$[k^{(2)}]$ is the initial displacement matrix of Marcal [25].

$[k^{(0)}]$ is the small displacement stiffness matrix.

The element stiffness matrices and the nodal equivalent forces are then summed in the usual direct stiffness manner to obtain master stiffness equations represented by equations in capitals

$$\Delta\{P\} = \left([K^{(1)}] + [K^{(2)}] + [K^{(0)}]\right)\Delta\{u\} \tag{9}$$

General Purpose Programs

The stiffness equations developed are quite general and are not restricted to a particular type of element. It is therefore possible to write a general purpose program which implements the theory. This program will then serve as a basic and common program to which subroutines may be added to account for specific characteristics belonging to the particular type of element (or element combinations) being used. Two general approaches to programming language have been adopted. The one most favored by the developers of ASKA, FORMAT, MAGIC, NASTRAN, SAMIS and STRUDL is to make use of some type of matrix interpretive language. Here the intention is to develop machine independent concepts and also lay the foundation for easy implementation of further theoretical developments. However, most of these programs have been developed under the influence of particular computers and programming languages, so that the aims of complete machine independence and freedom from bookkeeping requirements have not been fully achieved. On the other hand, there have been other programs (ELAS, MARC2) which were developed with FORTRAN as the programming language and making use of matrix support packages.

The one common feature to both approaches is the attempt to implement the matrix manipulation required by the theory in as general a form as possible. We see from (7) and (8) that the matrix operations required to form the stiffness matrices do not change. Similarly, the assembly of the element stiffness to form the master stiffness matrix does not change with different elements.

In order to focus on the advantages of general purpose programs, we shall now focus on the program MARC2 developed at Brown University. Most of the features found in this program can readily be included in other programs so that the points to be discussed can be thought to apply equally to all programs in general. This program was developed with the intention of carrying out the common matrix operations required to solve finite element problems with nonlinear material and/or geometric behavior. Because it was meant to be used in a research environment, it was organized with a view to minimize the coding required to implement new elements. This is made easier by the use of numerical integration to form the element stiffness matrices. Only four user subroutines are required to form the $[\alpha][B]$ quantities and specify the weighting functions required to perform the numerical integration. The general purpose program carries out the rest of the calculations based on input data. In particular, a subroutine has been developed to implement the incremental Prandtl-Reuss relations. Another subroutine integrates these relations through the thickness for a plate or shell when required. Various subroutines enable the assembled matrix equations to be solved by either the direct or iterative approach, as well as giving the option of an in-core assembly and out-of-core solution. This program simplifies combined elastic-plastic or creep and large displacement analysis by reducing the amount of additional programming required from a user. The nonlinear problem is converted to a series of piecewise linear problems. There is now an increase of an order of

magnitude in computing time required compared to a linear elastic solution since it takes about ten steps to trace the load history of a structure to its buckling or limit load.

Figure 1 shows the flow chart for the general purpose program MARC2. The procedures depicted in the main flow are the control, assembly, application of boundary conditions and the solution of the master stiffness equation. Two subroutines interface with the user subroutines and form the element stiffnesses and the initial stress and strain vectors respectively. Their purpose is to organize and perform the numerical integration to obtain the required quantities. In turn, these subroutines draw on the subroutines which form the linear incremental strain to stress transformation matrix [D] referred to above.

It is of interest to note here the various types of problems that can be handled by the program. It is noted that these can be performed with any combinations of elements and any combination of the following classes of problems:

1. Elastic
2. Elastic-plastic
3. Creep
4. Thermal strains
5. Large displacement
6. Large strains
7. Buckling (eigenvalue analysis at any load level)

We see immediately the advantages of using a general purpose program. Any feature implemented in the program can be combined with all previous developments. As illustrations of this we give examples of two recent additions to the general purpose program. The first was the implementation of a buckling analysis. Once this feature was checked it meant that it was possible to make use of all

previously developed elements with large displacement capabilities and perform buckling analysis of beams, plates and shells. Conversely, as an example of using the common features in the program, a new arbitrary, doubly curved shell element [36] was recently developed. It was then possible to use the element in the solution of all the seven classes of problems outlined above. This ability to preserve and exploit all previous developments is the main reason behind the impetus towards development of general purpose programs in recent years. This generalization is in accord with the development and use of computer programs in other areas. One other advantage of the general purpose program is that the main flow of this program will be used frequently and confidence in such a program will be more readily established.

There are also some drawbacks in such a general approach which are perhaps not so evident. First of all there is its slower running time because of the many conditional statements in the program. This slower speed is particularly noticeable in the larger computer systems where parallel computing devices are employed. Such a program also tends to become large, particularly if there is a large team working on it, and the limits of computer storage are quickly reached. Another problem to be overcome is that of verification documentation and dissemination of such a program. Because MARC2 was intended to be used in a research environment, the coding has been kept to a minimum. Even so, it has grown to about 6000 FORTRAN statements and already makes severe demands on new users. It is interesting to note here that it takes a new user about a month and a half to learn the program and begin to contribute to its development by modifying it. One major disadvantage in developing such a program is the difficulty in keeping changes made by one worker from interfering with the programming work of others.

Note on the Cost of Computing

In this note we shall examine the relationship between the size of a problem and the cost of computing. The actual cost of computing depends on the system configuration and it is possible to obtain differences in costs of up to factors of 2 by simply choosing different machines of the same nominal speed, as well as by choosing the same make of computer but using it at different installations. The important point to be recognized is that core is now required on the faster machines in such large sizes that its cost is as much as that of the central processor (C.P.U.). Thus realistic accounting procedures recognize this and, since particular system configurations are designed with certain functions in mind, its use for other purposes may cause a certain penalty. We shall use here the total system time as a measure of the computing required to solve each problem and not merely the C.P.U. time used. The size of a problem is dependent on the system configuration and will differ from one machine to the next. Thus the following discussion is an attempt to measure the relative cost of solving relatively large to small size problems on a particular computer. The ability to handle large problems is dependent on the size of core available to the user and, at the same time, it is also dependent on the core left for simultaneous use by other users. Because the core requirements for a finite element problem are determined by its master stiffness equation, we shall measure the size of the problem by the product of the number of degrees of freedom with the half-band-width. With this definition the size of a problem can be divided into three distinct categories which are determined by the peripheral storage required to solve the master stiffness equations. These solutions fall into the following categories, viz., the in-core assembly and solution, in-core assembly and out-of-core solution, and the out-of-core assembly and solution. In the first

category the complete solution can be effected in-core. In the second, the core requirements are such that assembly of the matrix can be performed in core by packing the matrix while the solution is carried out with the aid of magnetic discs or tapes. This course of action usually doubles the size that can be handled by an in-core solution. Finally, the third category is one in which the problem is so large that both assembly and solution must be performed out of core. Figure 2 shows a plot of total system time used against the size of problem. The limits of the in-core solution and the partial in-core solution are shown. Because of the relative speeds of C.P.U. to I./O. operations the total system time is shown increasing with increasing computer speeds. No attempt is made to show relative computing costs. Figure 2 also shows a curve for computing on a time-shared machine which is able to simulate practically unlimited core size for the user, the so-called virtual machines. It appears that operation on such a computer does not penalize a user unduly for working out of core since this is the normal mode of operation for which the system is planned. A distinct advantage in cost can be gained by solving the larger problems on such a machine. The writer's experience does bear this out.

The demands on the computer is not the only cost involved in a finite element analysis since the effort required for coding and the preparation of data must also be taken into account. The question of overall economics deserves further attention. It would be interesting, too, to combine this with further study of future hardware and software development of computer systems.

Case Studies

In this section we present a series of case studies which illustrate various facets of the current state-of-the-art of finite element analysis. It is hoped that these studies taken together will give an overall picture of the

progress made in this area. The writer has mainly drawn from results obtained in conjunction with his colleagues. Other choices could also have been made, but the present ones simply reflect greater familiarity with the results, as well as ready access to it.

1. Substructural Analysis of the 747 Aircraft Wing-Body Intersection [37]

This example of an elastic analysis is included to show the large-scale problems that are handled by the finite element method. The method of substructures or matrix partitioning is found to be the best way of reducing the problem to manageable proportions in both the data handling and the equation solving aspects of the problem. The substructures are shown in schematic form in Fig. 3. These were idealized by a combination of rod, beam, shear and constant strain elements. The whole problem resulted in 13,870 degrees of freedom. The substructuring reduced the largest band-width that had to be handled at any one stage. In connecting the substructures there were a total of 709 degrees of freedom that interacted at the interface. The effort that is involved in performing the analysis of the problem is described in [37]. The problem is restricted to linear elastic behavior; however, with the current rate of progress in the area, it is not difficult to envisage the same problem being solved with nonlinear material and geometric behavior.

It is of interest to note that about a hundred man-months of effort stretching over seven months was required. Much of the model idealization and work on the substructures proceeded in parallel. Twenty-eight hours of CDC 6600 C.P.U. time and one hundred and twenty hours of residency time was required for an error-free pass through the system.

2. Elastic-Plastic Analysis of Tensile Specimen with Semi-Elliptic Crack [38]

The next example is one of the analysis of a semi-elliptic crack in a tensile specimen using a combination of 648 three-dimensional cubic and isoparametric elements [38] and 900 nodal points or 2,700 degrees of freedom. Eight layers of elements were used to model the problem. Figure 4 shows the bottom layer of elements and the layout adopted to represent the semi-elliptic crack. The solution was carried out by first obtaining the load to cause the most highly stressed element to yield. Four increments, each equal to 0.07 of the elastic load were added to study the elastic-plastic behavior. The progress of plastic yielding in the second and fourth load increment is shown in Fig. 5. The elastic solution took 45 minutes of C.P.U. time on the IBM 360-91, and subsequent elastic-plastic increments took about 15 minutes per increment. A method of successive over-relaxation was used.

3. Analysis of Shell-Nozzle Junction with Combined Shell and Triangular Ring Elements [39]

This example is included to show a combination of shell and solid elements by the method of linear constraints [39]. A mild steel shell nozzle junction under pressure was studied experimentally by Dinno and Gill [40]. This same problem was analyzed using the mesh in Fig. 6. Triangular ring elements are used in and around the weld section, and shell elements are used throughout the main body of the shell and nozzle. Comparison of the finite element results with experimental data is shown in Fig. 4. The actual differences between the peak stresses can be seen in Table 1. The hybrid finite element results show considerable improvement over a previous modified shell theory approach using a band of pressure for the junction [41]. That theory was itself a large improvement over the simple shell theory.

	LIMIT OF PROPORTIONALITY (lb/in^2
Experimental [40]	800
Simple shell theory [41]	340
Band theory [41]	630
Hybrid analysis	793

TABLE 1. First Yield of Shell-Nozzle Junction with Internal Pressure

4. <u>Imperfect Hemisphere under External Pressure [42]</u>

This example shows the combined effect of nonlinear geometric and material behavior. Both of these act together to drastically weaken the load resistance of the structure. The oblate spherical shell of Fig. 8 was analyzed by Bushnell [43] in the nonlinear elastic region. It was there observed that high "elastic" stresses were observed at the crown of the shell prior to collapse. This shell was analyzed with a dimensionless yield stress of 0.00666 E together with a linear work-hardening curve with a slope of 0.05 E. This corresponds roughly to the stress strain curve of an Aluminum Alloy.

Figure 9 gives a comparison between the buckling pressure of the elastic shell of [43] and the present elastic-plastic results. The results are plotted in terms of the parameters used in [43]. The classical buckling load ρ_c is defined by

$$\rho_c = 1.21(2H/R)^2 E$$

where E is the Young's Modulus

R is the radius of the sphere

and H is the half-thickness.

The geometric parameter λ is defined by

$$\lambda = \sqrt{12(1-\nu^2)}\left(\frac{R}{2H}\right)^{1/2}\frac{R}{R_{imp.}}$$

where $R_{imp.}$ = mean radius of the oblate portion of the sphere, ν = Poisson's Ratio.

Plastic yielding has a considerable effect on the behavior of the oblate shells under external pressure. This effect increases with the thickness to radius ratio. It is noted that the failures at the higher thickness to radius ratios $(\lambda \leq 1.5)$ are due to membrane yield.

Reasonable agreement is also obtained with the elastic results of Bushnell [43] for the thinner shells which do not yield before buckling.

5. Infinite Incompressible Log [44]

The infinite log under symmetric line loading is shown in Fig. 10. This problem was solved by Oden [44] using the generalized Newton-Raphson method. The problem was reduced to 22 simultaneous equations by taking advantage of symmetry. For illustration purposes, the line loading P was taken to be 200 lb/in and for the Mooney constants $C_1 = 43.75$ lb/in^2 and $C_2 = 6.25$ lb/in^2 were used. Results in the form of the deformed profile are indicated in Fig. 11.

Conclusions and Future Work

In this paper we have examined the formulation and the implementation of a theory for nonlinear finite element analysis. The general purpose program was shown to be a versatile and flexible method of implementing the basic theory. It was found possible to classify three basic sizes of problems which were dependent on the ability of the computer to either assemble or solve the master stiffness matrix in core.

Case studies were given to illustrate representative applications of the theory. If, as is argued here, the general purpose program is the key to widespread applications of the theory, then much more remains to be learnt about its development and organization. Little is known about the best way to match programs with particular computer system configurations. Even less is known about the impact of future hardware developments. Finally, rigorous procedures have yet to be developed for verification of these programs.

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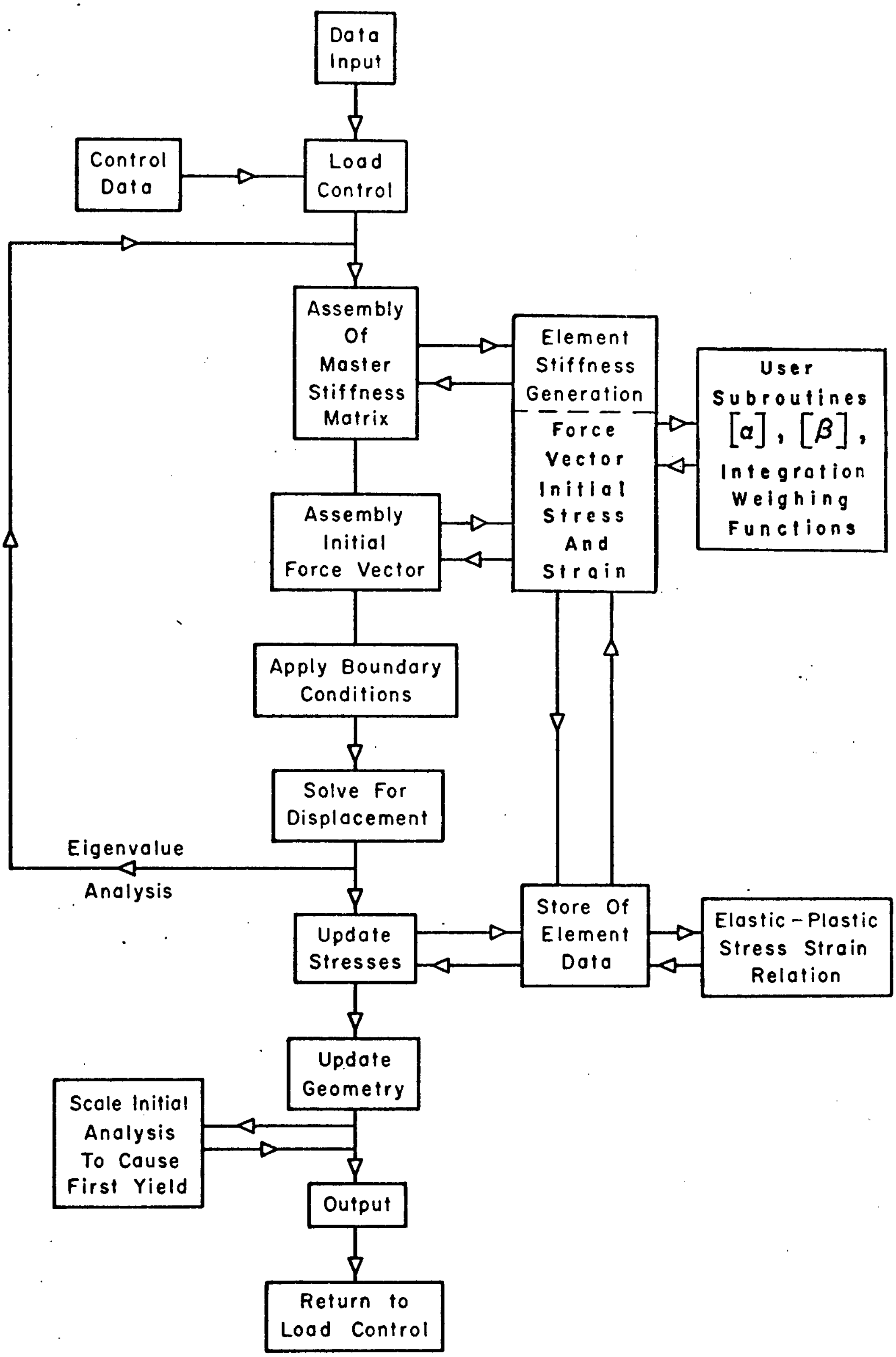


FIG. I FLOW CHART FOR COMPUTER PROGRAM

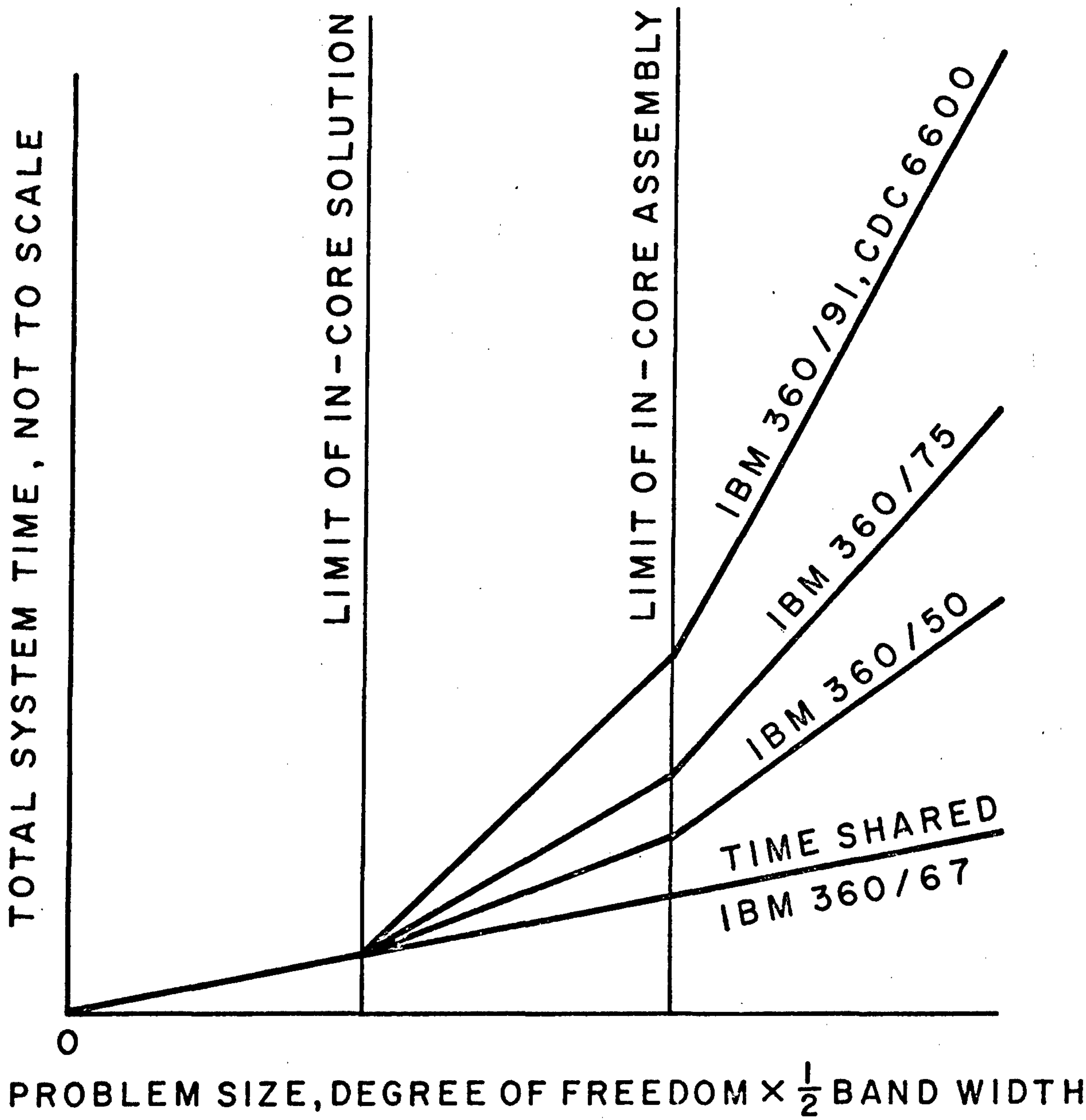


FIG. 2 COMPUTING TIME – PROBLEM SIZE

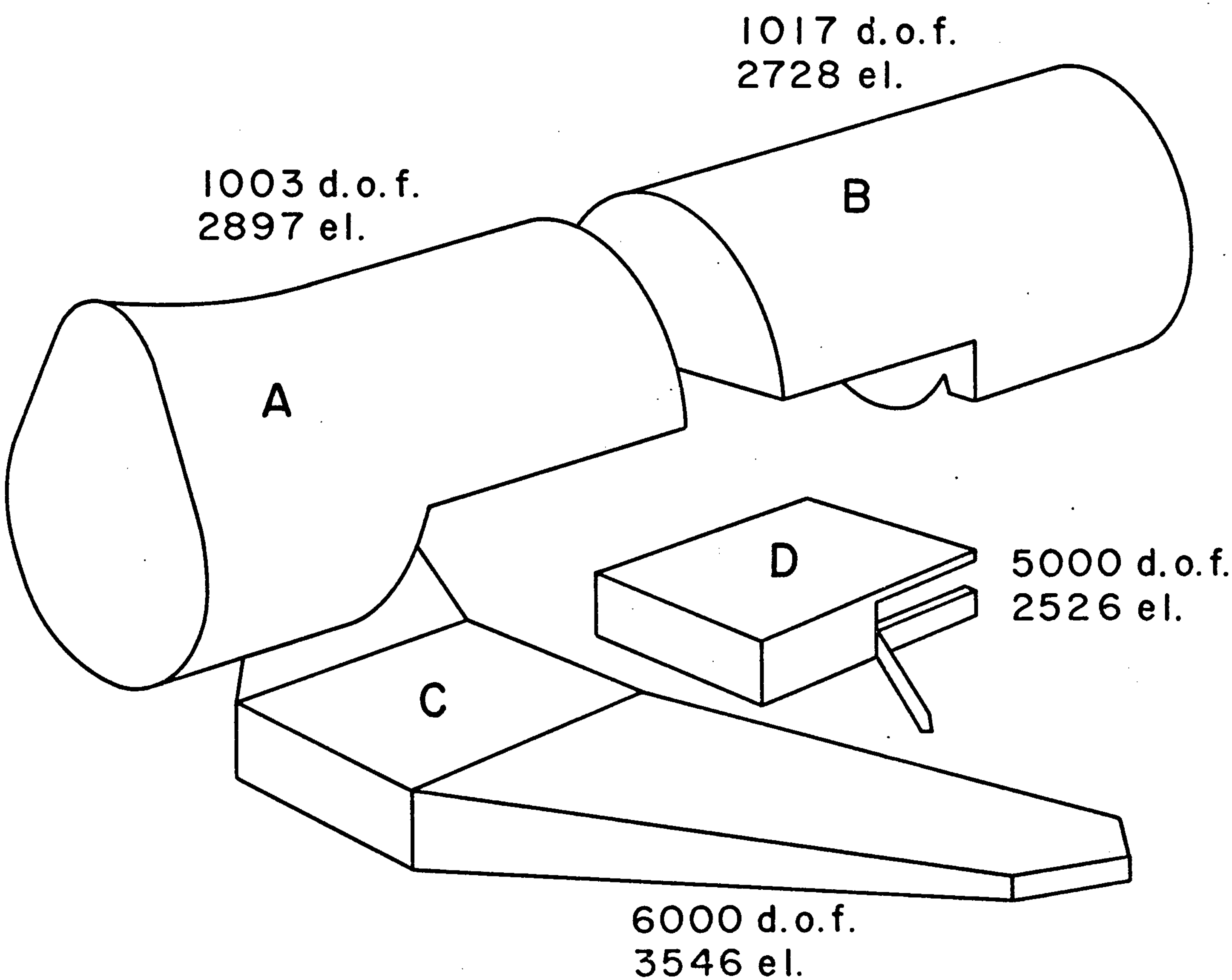


FIG. 3 SCHEMATIC OF SUBSTRUCTURES FOR BOEING 747

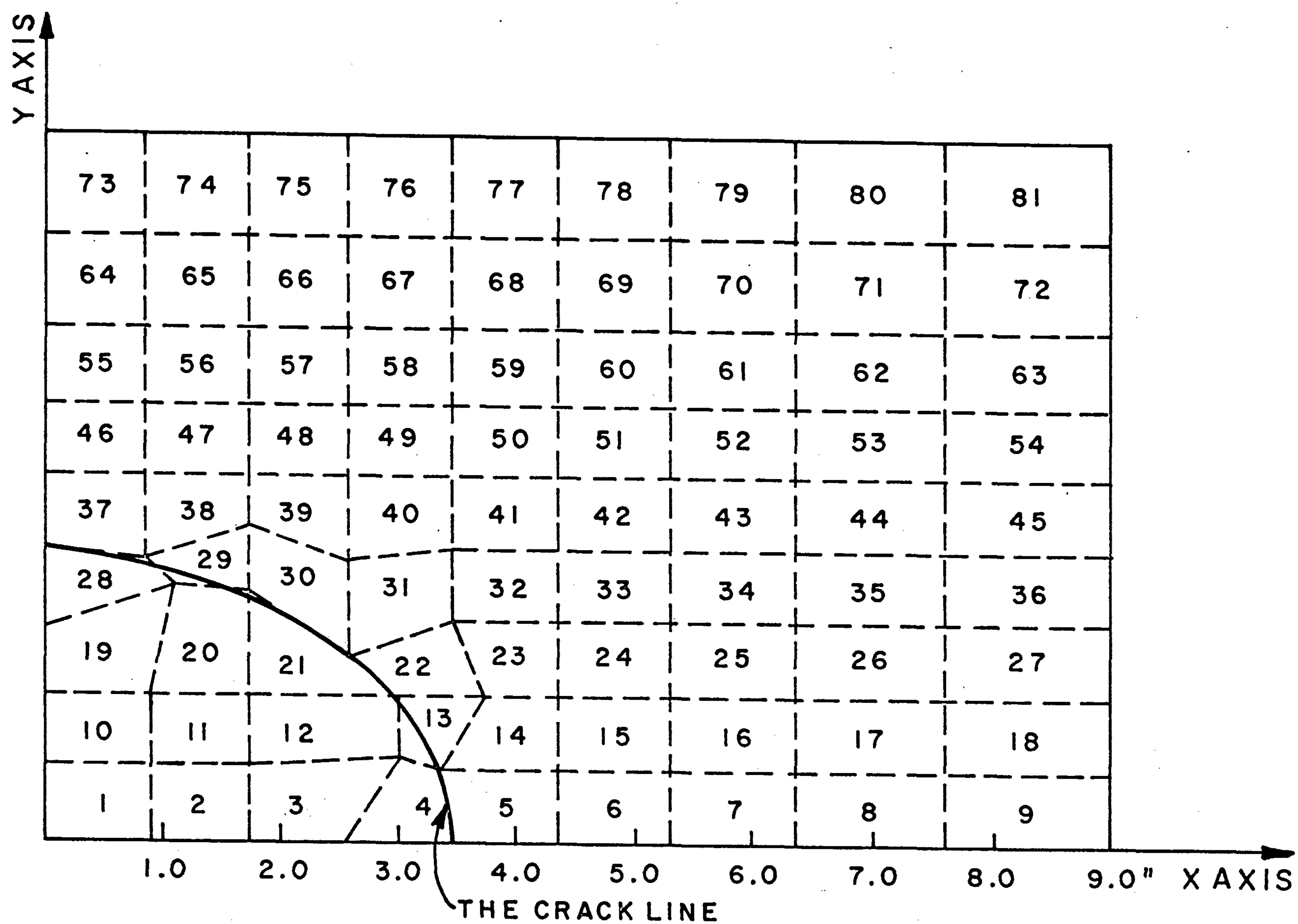
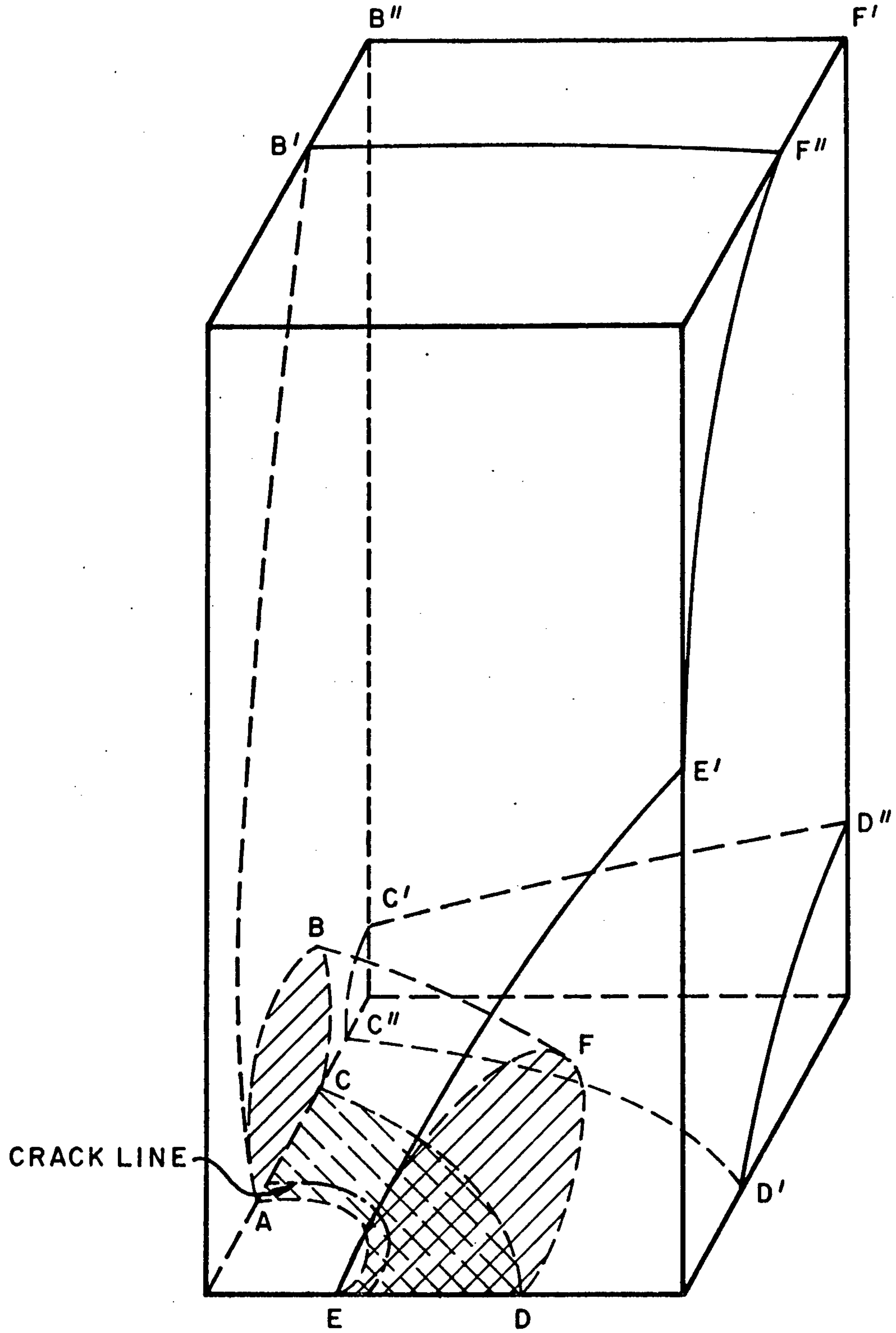


FIG. 4 ELEMENTS IDENTIFICATION IN THE CRACK PLANE



1) PLASTIC ZONE ABCDEF IS AFTER TWO INCREMENTS
2) PLASTIC ZONE AB'B''C'C''D'D''F'F''E'E IS AFTER 4TH INCREMENT

FIG. 5 PROGRESS IN PLASTIC YIELDING TENSILE PLATE WITH SEMI-ELLIPTIC CRACK

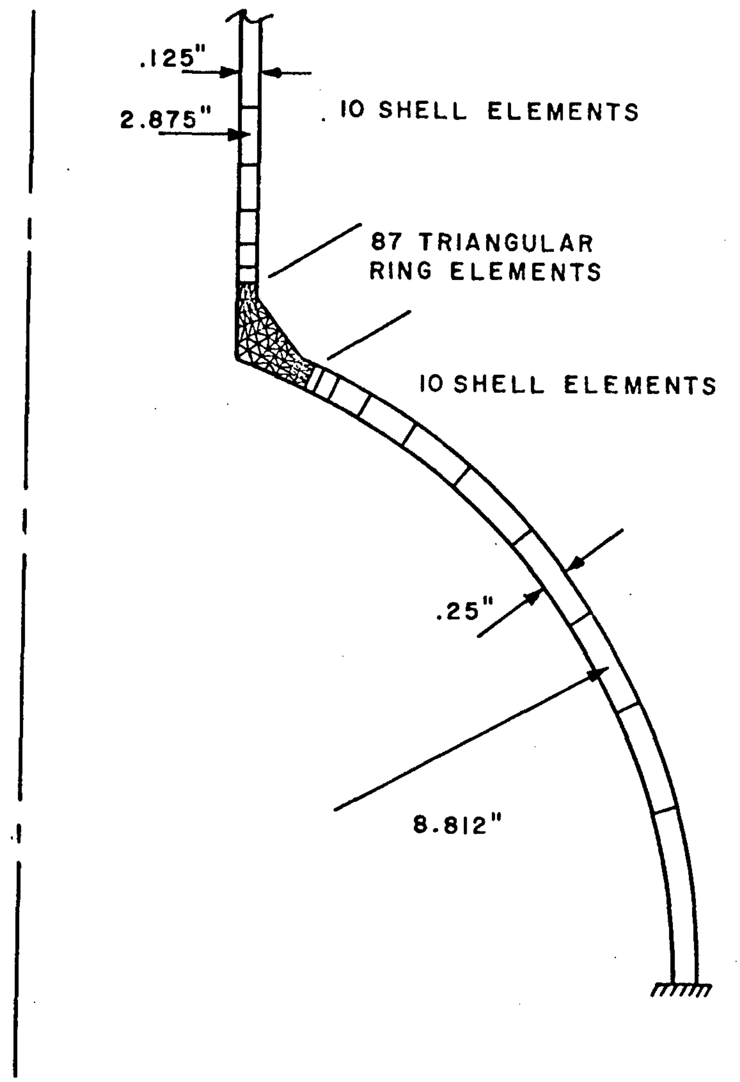


FIG. 6 MESH FOR SHELL-NOZZLE JUNCTION. (NOT TO SCALE)

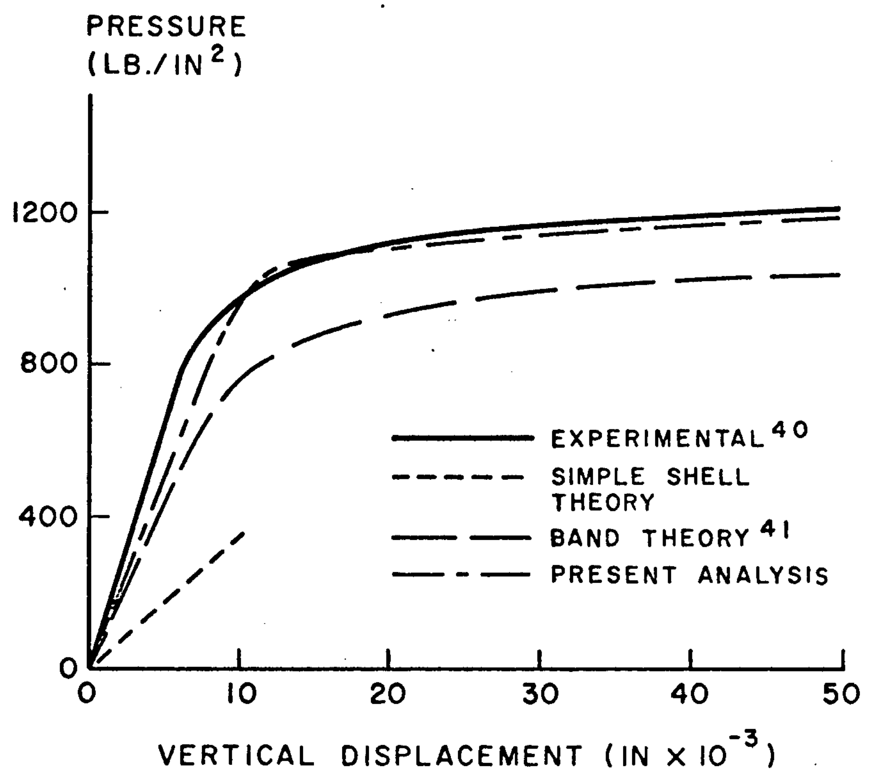


FIG. 7 NOZZLE DISPLACEMENT

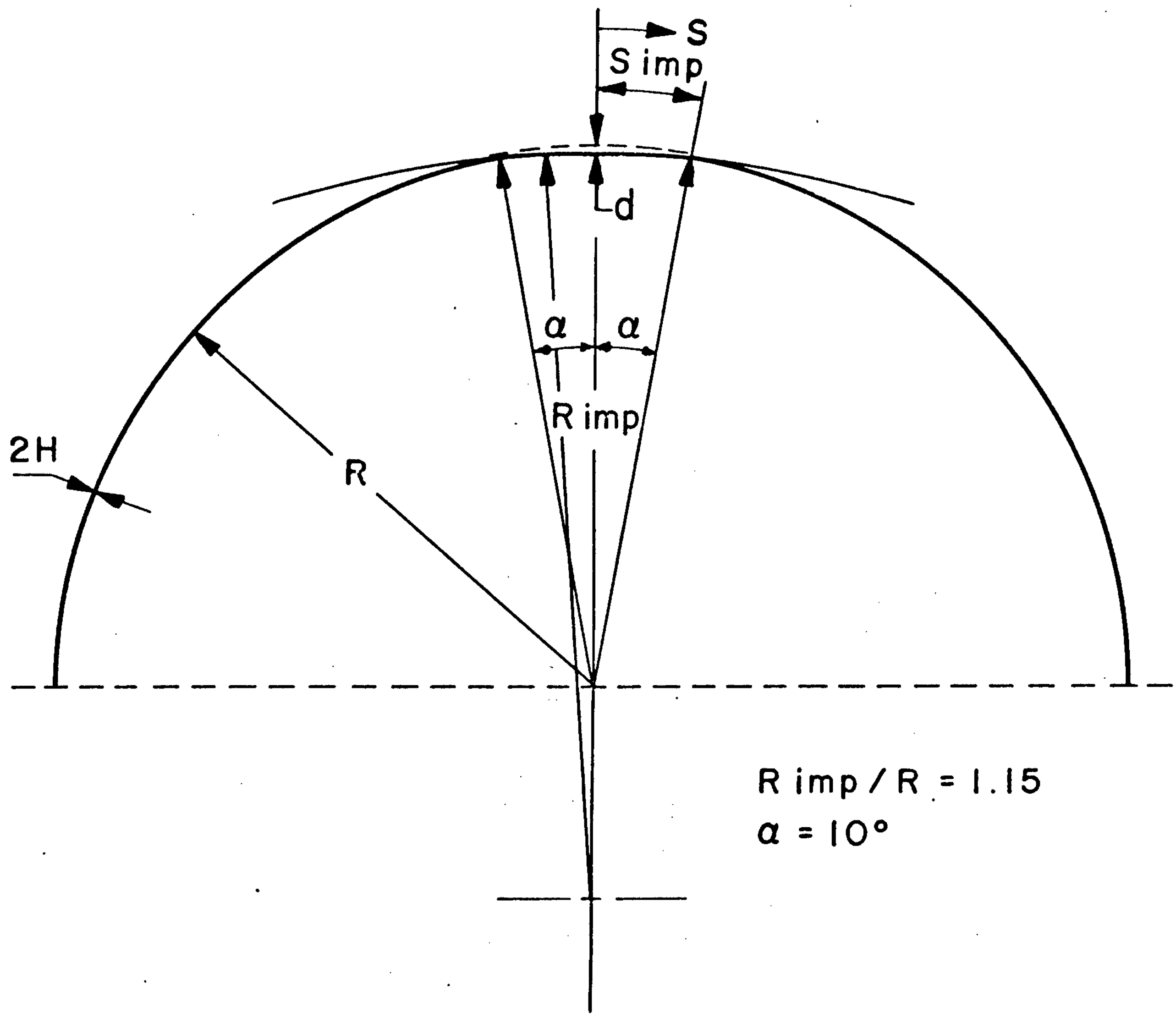


FIG. 8 EXTERNALLY PRESSURIZED IMPERFECT HEMISPHERE.

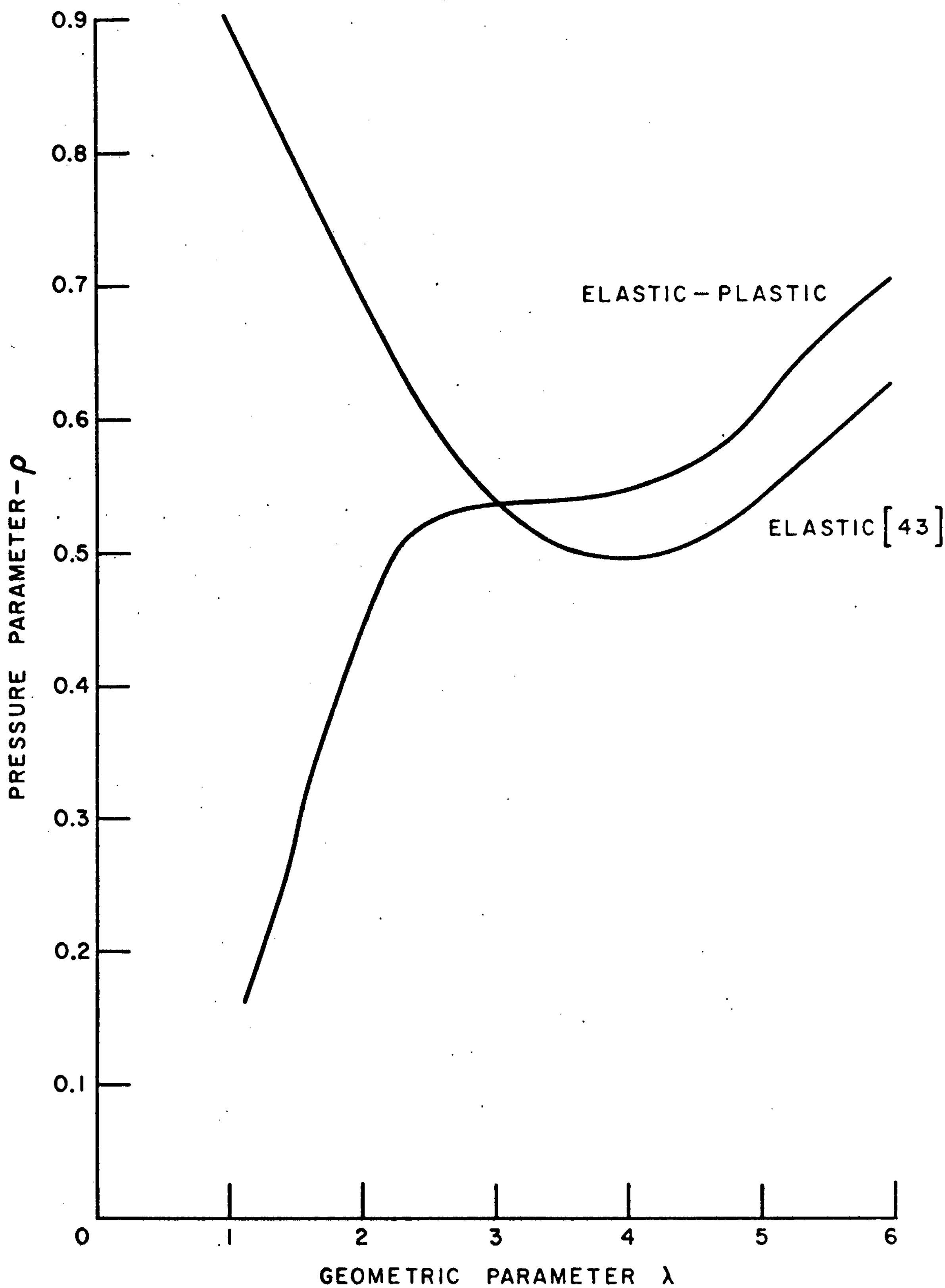


FIG.9 BUCKLING PRESSURES FOR OBLATE SHELLS, Rimp/R=1.15

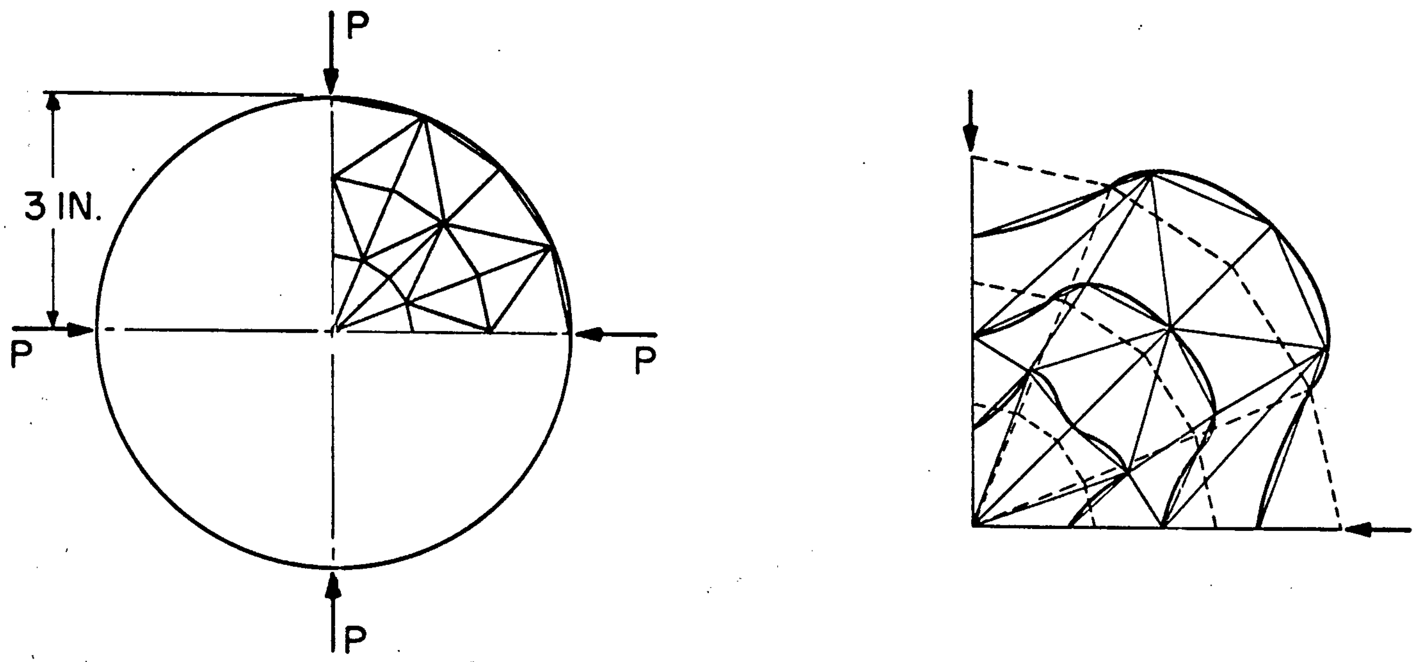


FIG. 10 FINITE INCOMPRESSIBLE LOG